Lecture 12. ***Parametrical synthesis of nonlinear laws of control using a direct method of Lyapunov***

Let's the following linear ACS be given:

$$\dot{x} = Ax + Bu, \qquad (6.11)$$

where $x \in R^n, u \in R^1, A, B$ are matrixes of constant coefficient of corresponding dimensions. We choose the control law:

$$u = -mx + u_0, \qquad (6.12)$$

where u_0 is an additional signal, *m* is a vector of control law parameter.

By closing the given system with the chosen control law

$$\dot{x} = Ax - Bmx + Bu_0 = Cx + Bu_0,$$

where $C = (A - Bm)$, it is possible to insert some definite properties in a closed system.

It is possible, when amplitude of controlling signal is restricted:

$$|u_i| \le u_{i_{\max}}, i = \overline{1, l}.$$

In a lot of situations nonlinear synthesis allows to improve dynamical characteristics of the linear system.

Example 6.16. Let system be given by equations (6.11), (6.12).

The roots of characteristic equation are complexly conjugate α < 0; let's $x \in R^2$. There are shown (fig. 1) portraits of linear and nonlinear systems. It is obvious, that a nonlinear system is faster, than a linear one.

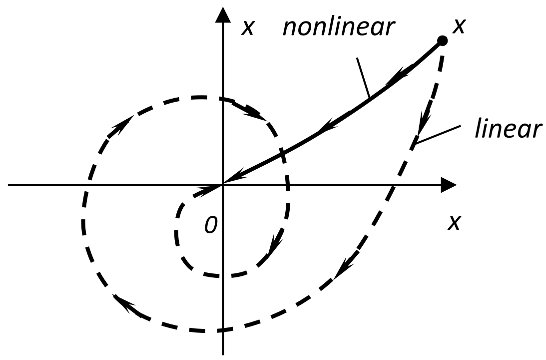

Fig. 1. Phase portrait

Let us solve a problem of parametric synthesis of nonlinear ACS, by means of Lyapunov's second (direct) method.

Along with equation (6.11) we will consider transient system:

$$\dot{x}^T = x^T A^T + u^T B^T. \tag{6.13}$$

Let matrix A of system (6.11) be stable, which means that the own matrix values have negative real parts:

$$Re\ \lambda_i(A) < 0,\ i = \overline{1,n}.$$

Let us choose Lyapunov's function in a square-law form as the following:

$$V(x) = x^T P x,$$

where $P = P^T$, $P > 0$, i.e. the matrix P is a symmetric and positively defined matrix.

The full differential Lyapunov's function is:

$$\frac{dV}{dt} = \dot{x}^T P x + x^T P \dot{x}. \tag{6.14}$$

Substitute (6.11) and (6.13) in (6.14). Then the full differential of Lyapunov's function will be equal to the following:

$$\begin{aligned}\frac{dV}{dt} &= (x^T A^T + u^T B^T)Px + x^T P(Ax + Bx) = \\ &= x^T A^T Px + u^T B^T Px + x^T PAx + x^T PBu = \\ &= (x^T A^T Px + x^T PAx) + (u^T B^T Px + x^T PBu) = \\ &= -x^T Qx + 2u^T B^T Px = -x^T Qx + 2u^T (Cx)\end{aligned},$$

where $C = B^T P$, $Q = Q^T$ is a symmetric matrix, $Q > 0$ is of fixed positive sign.

Hence, the full differential of Lyapunov's function is equal to the following:

$$\frac{dV}{dt} = -x^T Qx + 2u^T (Cx). \tag{6.15}$$

Choose $u = u_{max} sign\ (Cx)$.

At such a choice of the control function the second item in equation (6.15) will be always negative (the first is negative-sign).

Hence $\frac{dV}{dt} < 0$, that provides *asymptotical stability to the system according to Lyapunov.*

Thus, the procedure of parametric synthesis of nonlinear ACS is broken into two stages:

I. It is necessary to solve Lyapunov's matrix equation of the following type:

$$A^T P + PA = -Q.$$

While setting $Q = Q^T > 0$, we find the solution of matrix equation, i.e. matrix P, when $P = P^T$; $P > 0$.

II. It is necessary to define matrix C, equal to

$$C = B^T P.$$

Below there are represented (fig. 2) matrix structural scheme of a system with the given type of nonlinearity (system 1).

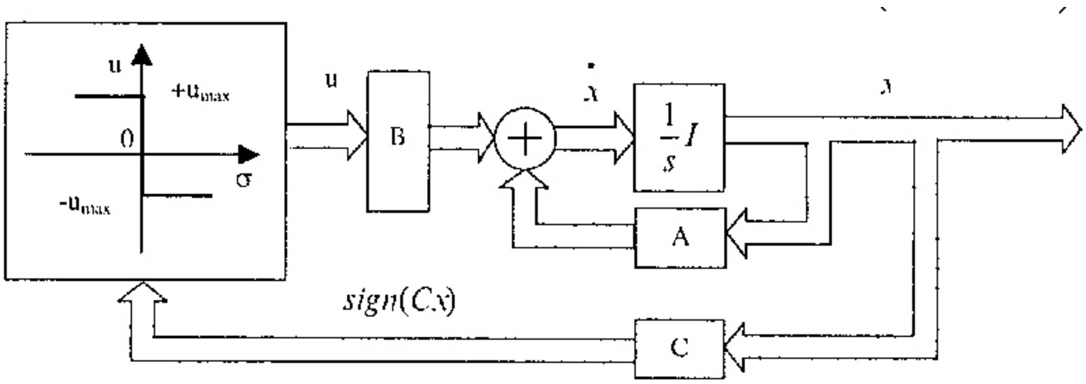

Fig. 2. Matrix structural scheme of system 1

Each component of the control vector must pass through nonlinearity of such type. That is why the system must have *l* relay characteristics.

The same approach allows realizing other control laws.

For example: $u = -\,sat\,(Cx)$,

where *sat* is the function of saturation, i.e. the other type of nonlinearity fig. 3 (system 2).

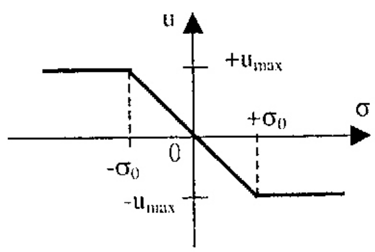

Fig. 3. Nonlinearity with saturation

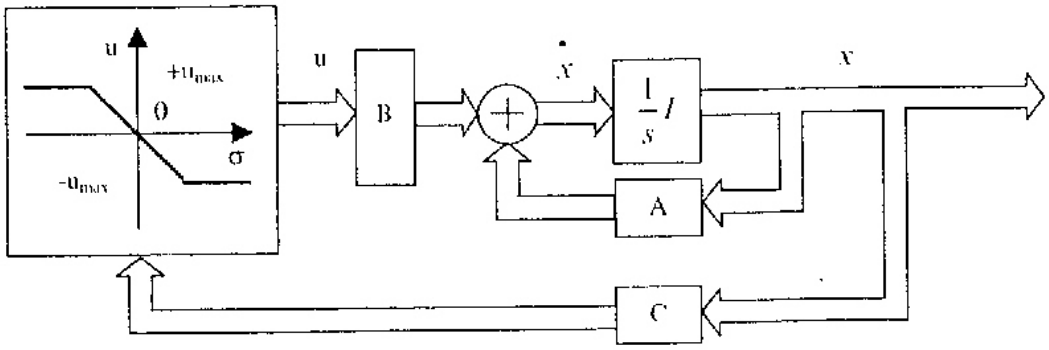

Fig. 4. Matrix structural scheme of system 2

The reaction of the second system is a bit slowed-down in comparison with the first system, i.e. the speed is less, but overshoot in the *2nd* system is less, than in the first one, which visually shows transients of the system (fig. 5).

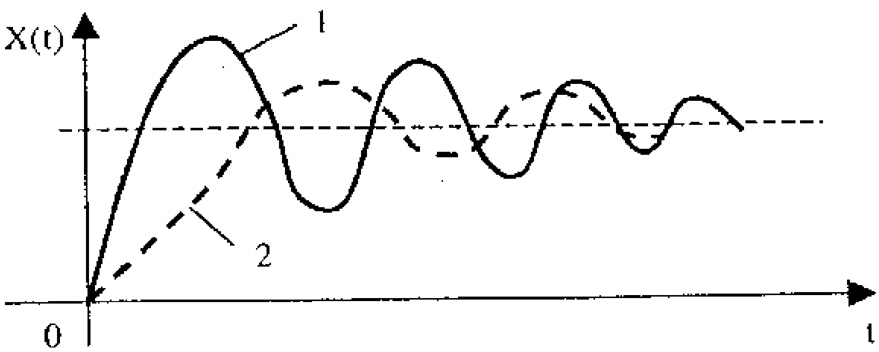

Fig. 5. Transient processes of systems 1 & 2